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# The Product Formula and Convolution Structure Associated with the Generalized Hermite Polynomials

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Here the product formula for the generalized and suitably normalized Hermite polynomials with parameter $\mu \geqslant 0$ will be explicitly established. Its measure turns out to be absolutely continuous and supported on two disjoint intervals lying symmetrically on the real line, provided that $\mu > 0$. In the limit case $\mu = 0$, which is associated with the classical Hermite polynomials, four additional point masses occur at the endpoints of the two intervals. As an application, the product formula is used to introduce a generalized translation operator and a corresponding convolution product on appropriately weighted Lebesgue spaces. To this end, norm estimates of the translation operator from above and below are presented. For any $\mu \geqslant \frac{1}{2}$, this gives rise to a quasi-positive convolution algebra. © 1993 Academic Press, Inc.

## 1. INTRODUCTION AND DESCRIPTION OF RESULTS

Product formulas are deep and very useful identities associated with a discrete or continuous function system. Roughly speaking, such a formula represents the product of any two values of an orthogonal polynomial, say, in terms of a Stieltjes integral which depends linearly on the polynomial itself. This is the basic information needed to introduce a generalized translation operator in the sense of J. Delsarte [5] and B. M. Levitan [14] and to define a convolution product on a function space which usually becomes a Banach algebra under this operation. Such a convolution structure then plays the same fundamental role in the harmonic analysis of the corresponding orthogonal expansion as ordinary convolution does in Fourier analysis.

Moreover, in those cases in which a positive measure is induced by the product formula, it may also be used to establish a convolution algebra of

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bounded Borel measures on a locally compact space. This is the concept of the so-called hypergroups which form an abstract, but natural framework for developing a harmonic analysis; for definitions and examples see, e.g., H. Heyer [11] and the literature cited there.

Prominent and well-known examples of product formulas are those for the Bessel functions and for the ultraspherical polynomials due to Gegenbauer; cf. Watson [19, 11.41, 11.5]. Almost a century later, G. Gasper [7, 8] and T. H. Koornwinder [12] independently extended these identities to the more general Jacobi polynomials and established the corresponding convolution structure; cf. also [3, 17]. Gasper's result, in turn, was employed by his student Th. P. Laine [13] in order to derive the product formula for the generalized Chebyshev polynomials. As to the Laguerre polynomials and series, a satisfactory convolution structure was introduced by E. Görlich and the author in 1982 [10]; cf. also [15]. Finally it should be mentioned that very recently, W. C. Connett, A. L. Schwartz, and the author [2] found the product formula of the spheroidal wave functions.

The Hermite polynomials $H_n(x)$, $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, are known to be the polynomials orthogonal on the real line with respect to the weight function $\exp(-x^2)$. Surprisingly, they are the only classical orthogonal polynomials for which the product formula has apparently not been treated or even stated as yet. Likewise, the notions of a generalized translation and a convolution associated with Hermite series have remained nebulous or vague so far. One purpose of this paper is to close this gap.

Actually, we will deal with the more general, but less known "generalized" Hermite polynomials $H_n^{(\mu)}$, $n \in \mathbb{N}_0$, which were introduced for any $\mu > -1/2$ by G. Szegő [18, Problem 25] and studied in greater detail by T. Chihara [1, V2]. They satisfy the orthogonality relation ($[a]$ = largest integer less than or equal to $a$)

$$\int_{-\infty}^{\infty} H_n^{(\mu)}(x)\, H_m^{(\mu)}(x)\, e^{-x^2}\, |x|^{2\mu}\, dx = h_n^{(\mu)}\, \delta_{n,m}$$

with (1.1)

$$h_n^{(\mu)} = 2^{2n}\,\Gamma\left(\left[\frac{n}{2}\right]+1\right)\Gamma\left(\left[\frac{n+1}{2}\right]+\mu+\frac{1}{2}\right)$$

and thus include the classical Hermite polynomials for $\mu = 0$. Due to the uniqueness of the Gram–Schmidt orthonormalization process, the generalized Hermite polynomials are explicitly given in terms of the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} R_n^{(\alpha)}(x) := \binom{n+\alpha}{n}\, {}_1F_1(-n; \alpha+1; x), \tag{1.2}$$

$n \in \mathbb{N}_0$, $\alpha > -1$, by

$$H_n^{(\mu)}(x) = \begin{cases} (-1)^k\, 2^{2k} k!\, L_k^{(\mu - 1/2)}(x^2) & \text{if } n = 2k, \\ (-1)^k\, 2^{2k+1} k!\, x L_k^{(\mu + 1/2)}(x^2) & \text{if } n = 2k+1, \end{cases} \quad k \in \mathbb{N}_0. \tag{1.3}$$

Hence, $H_n^{(\mu)}(x)$ is an even or odd function according as $n$ is even or odd. We also use the normalization

$$\tilde{H}_n^{(\mu)}(x) = \begin{cases} H_{2k}^{(\mu)}(x)/H_{2k}^{(\mu)}(0) = R_k^{(\mu - 1/2)}(x^2) & \text{if } n = 2k \\ H_{2k+1}^{(\mu)}(x)/(H_{2k+1}^{(\mu)})'(0) = x R_k^{(\mu + 1/2)}(x^2) & \text{if } n = 2k+1 \end{cases} \tag{1.4}$$

with orthonormal constants

$$\tilde{h}_n^{(\mu)} := \int_{-\infty}^{\infty} [\tilde{H}_n^{(\mu)}(x)]^2\, e^{-x^2} |x|^{2\mu}\, dx$$

$$= \begin{cases} \Gamma(\mu + 1/2)\, k!/(\mu + 1/2)_k & \text{if } n = 2k \\ \Gamma(\mu + 3/2)\, k!/(\mu + 3/2)_k & \text{if } n = 2k+1. \end{cases}$$

Our first main result, Theorem 2.1, presents the product formula of the generalized Hermite polynomials for any $\mu > 0$ in the form

$$\tilde{H}_n^{(\mu)}(x)\, \tilde{H}_n^{(\mu)}(y) = \int_{-\infty}^{\infty} \tilde{H}_n^{(\mu)}(z)\, dm_{x,y}^{(\mu)}(z), \tag{1.5}$$

where $x, y \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $m_{x,y}^{(\mu)}$ denotes a real Borel measure not depending on $n$. This measure turns out to be absolutely continuous, so that there is an explicitly given kernel function $K_H^{(\mu)}(x, y, z)$ satisfying

$$dm_{x,y}^{(\mu)}(z) = K_H^{(\mu)}(x, y, z)\, e^{-z^2} |z|^{2\mu}\, dz. \tag{1.6}$$

From the representation of the kernel, namely (2.8), all of its important features can be read off. First, as a function of $z$, the kernel is supported on the union of two disjoint intervals, namely

$$S(x, y) = (-|x| - |y|, -||x| - |y||) \cup (||x| - |y||, |x| + |y|) \tag{1.7}$$

for any $x$, $y \in \mathbb{R}$. Obviously, the support satisfies the two symmetry properties

$$S(x, y) = S(y, x), \qquad z \in S(x, y) \Leftrightarrow x \in S(z, y).$$

Therefore and since, in principle, $K_H^{(\mu)}(x, y, z)$ is a function of $\Delta(x, y, z)$ alone, where $\Delta$ is defined in (2.4) below as twice the area of a plane triangle of sides $x, y, z$, the kernel is moreover a symmetric function of all three variables $x, y, z \in \mathbb{R}$; this is in accordance with its formal series representation based on the orthogonality (1.1),

$$K_H^{(\mu)}(x, y, z) \sim \sum_{n=0}^{\infty} [\tilde{h}_n^{(\mu)}]^{-1} \tilde{H}_n^{(\mu)}(x) \tilde{H}_n^{(\mu)}(y) \tilde{H}_n^{(\mu)}(z). \tag{1.8}$$

Finally, $K_H^{(\mu)}(x, y, z)$ is given in terms of the normalized Bessel function of the first kind,

$$\mathscr{J}_\nu(z) = 2^\nu \Gamma(\nu+1) z^{-\nu} J_\nu(z) := {}_0F_1(-; \nu+1; -z^2/4), \qquad \nu > -1. \tag{1.9}$$

But since the Bessel function is known to have (infinitely many) positive zeros, it is clear that the measure $m_{x,y}^{(\mu)}$ cannot be positive for any $\mu$; cf. Corollary 2.4.

The case of the classical Hermite polynomials, $\mu = 0$, behaves somewhat differently from those where $\mu > 0$, since the limit of the right-hand side of the product formula (2.7), as $\mu$ tends to $0+$, exists only in the distributional sense. As is stated in Theorem 2.3, (2.9), the measure $m_{x,y}^{(0)}$ now consists of two absolutely continuous components together with four point masses at the endpoints.

Methodically, we proceed from the relationship (1.3) between the generalized Hermite and Laguerre polynomials and employ the product formula of the Laguerre polynomials for any parameter $\alpha \geqslant -1/2$; see (2.3)–(2.4). Another important step is the "symmetrization" of the support of the required formula, where we take advantage of an idea used by Laine [13] in the somewhat related case of the generalized Chebyshev polynomials.

Quite often in special function theory, it is much harder to obtain a particular result—especially when arising in a non-typical limiting case—than to prove a more general result depending on a number of parameters. Usually the parameters provide the degree of freedom which is needed to see useful relations or analogies. Our problem concerning the classical versus generalized Hermite polynomials is an example in this respect. In fact, the additional parameter $\mu$ of the generalized Hermite polynomials allows one to define a continuum of weighted Lebesgue spaces to which the product formula can be suitably extended, namely

$$L_\mu^p = \{f \text{ measurable on } \mathbb{R};\ \|f\|_{p,\mu} < \infty\}, \qquad 1 \leqslant p \leqslant \infty, \tag{1.10a}$$

where

$$\|f\|_{p,\mu} = \begin{cases} \{\int_{-\infty}^{\infty} |f(x)\, e^{-x^2/2}|^p\, |x|^{2\mu}\, dx\}^{1/p}, & p < \infty \\ \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)\, e^{-x^2/2}|, & p = \infty. \end{cases} \tag{1.10b}$$

Let the "generalized Hermite translation" be given by

$$T_y^{(\mu)}(f;x)=\begin{cases}\int_{-\infty}^{\infty} f(z)\,dm_{x,y}^{(\mu)}(z), & xy\neq 0\\ \tfrac{1}{2}[f(x)+f(-x)], & y=0,\quad x\in\mathbb{R},\text{ a.e.}\\ \tfrac{1}{2}[f(y)+f(-y)], & x=0,\quad y\in\mathbb{R},\text{ a.e.,}\end{cases}\tag{1.11}$$

which clearly satisfies the characteristic property

$$T_y^{(\mu)}(\tilde{H}_n^{(\mu)};x)=\tilde{H}_n^{(\mu)}(x)\,\tilde{H}_n^{(\mu)}(y).\tag{1.12}$$

Then it turns out that $T_y^{(\mu)}$ is a bounded linear operator from $L_\mu^p$, $1\leqslant p\leqslant\infty$, into itself, provided that $\mu\geqslant 1/2$. For $0\leqslant\mu\leqslant 1/2$, however, the domain has to be further restricted. This will be proved in Section 3 as a consequence of respective estimates of the $L_\mu^1$ norm of the kernel $K_H^{(\mu)}(\circ, y, z)$; see Theorem 3.3.

The next step is to introduce the convolution product in the usual way by

$$(f*g)(y)=\int_{-\infty}^{\infty} T_y^{(\mu)}(f;x)\,g(x)\,e^{-x^2}\,|x|^{2\mu}\,dx.\tag{1.13}$$

As to the function spaces for which this is defined, we have to distinguish again between the cases $\mu\geqslant 1/2$ and $0\leqslant\mu<1/2$. In the first case, the norm estimate of the translation ensures, for instance, the existence of some constant $M_\mu\geqslant 1$ for which

$$\|f*g\|_{1,\mu}\leqslant M_\mu\,\|f\|_{1,\mu}\,\|g\|_{1,\mu},\tag{1.14}$$

while in the second case, the convolution (1.13) exists at least if $f, g\in L_\mu^1\cap L_{\mu/2+1/4}^1$; cf. Theorem 3.5. Furthermore, it follows by standard methods that the product (1.13) possesses the usual properties of a convolution such as commutativity, associativity and the fact that the generalized Hermite transform

$$f^\wedge(n)=\int_{-\infty}^{\infty} f(y)\,\tilde{H}_n^{(\mu)}(y)\,e^{-y^2}\,|y|^{2s}\,dy,\qquad n\in\mathbb{N}_0,$$

satisfies the fundamental identity

$$(f*g)^\wedge(n)=f^\wedge(n)\,g^\wedge(n).$$

Altogether, we obtain for any $\mu\geqslant 1/2$ a convolution structure for the generalized Hermite series

$$f(x)\sim\sum_{n=0}^{\infty} f^\wedge(n)\,\tilde{H}_n^{\mu}(x)\,[\tilde{h}_n^{(\mu)}]^{-1},$$

which is quasi-positive in the sense of Gasper [8]. Moreover, the constant $M_\mu$ in inequality (1.14) may be removed by rescaling.

At this point let us remark that in view of the "discontinuity" of the support (1.7), the measure $m_{x,y}^{(\mu)}$ of the product formula does not induce a hypergroup structure. This is not surprising, since it has recently been observed by W. C. Connett and A. L. Schwartz [4] that the Jacobi polynomials are virtually the only orthogonal polynomials which give rise to a one-dimensional (continuous) hypergroup; for a detailed proof and discussion of this phenomenon see [3]. Very recently, O. Gebuhrer [9] and also Hm. Zeuner proposed to weaken the canon of axioms of a hypergroup by dropping the support condition, but still requiring the positivity of the convolution. However, as mentioned above, the positivity condition is also violated in the present case of generalized Hermite polynomials. Nevertheless we believe that the results of this paper provide another interesting example (or counterexample) in this context which enriches the theory and may stimulate further discussions on the matter.

## 2. The Product Formula

For convenience we pass over from the Laguerre and Hermite polynomials to the respective functions (cf. (1.2), (1.4))

$$\begin{aligned} \mathscr{L}_n^{(\alpha)}(x) &= \exp(-x^2/2)\, R_n^{(\alpha)}(x^2), \qquad x \geq 0, \\ \mathscr{H}_n^{(\mu)}(x) &= \exp(-x^2/2)\, \tilde{H}_n^{(\mu)}(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{2.1}$$

so that the relationship (1.3) becomes

$$\mathscr{H}_n^{(\mu)}(x) = \begin{cases} \mathscr{L}_k^{(\mu-1/2)}(|x|), & n = 2k \\ x\, \mathscr{L}_k^{(\mu+1/2)}(|x|), & n = 2k+1. \end{cases} \tag{2.2}$$

As a basic tool we need the following kernel version of the Laguerre product formula [10; 15; 16; 17, (2.12)]. For $\alpha > -1/2$ and $x, y > 0$,

$$\mathscr{L}_n^{(\alpha)}(x)\, \mathscr{L}_n^{(\alpha)}(y) = \int_0^\infty \mathscr{L}_n^{\alpha}(z)\, \mathscr{K}_L^{(\alpha)}(x, y, z)\, z^{2\alpha+1}\, dz \tag{2.3a}$$

with

$$\mathscr{K}_L^{(\alpha)}(x, y, z) = \begin{cases} C_\alpha \dfrac{\Delta^{2\alpha-1}\, \mathscr{J}_{\alpha-1/2}(\Delta)}{(xyz)^{2\alpha}}, & |x-y| < z < x+y \\ 0, & \text{elsewhere.} \end{cases} \tag{2.3b}$$

Here, $C_\alpha = \Gamma(\alpha+1)/[\Gamma(\alpha+1/2)\,\Gamma(1/2)]$, $\mathcal{J}_\nu$ denotes the normalized Bessel function defined in (1.9), and

$$\begin{aligned}\Delta = \Delta(x, y, z) &= \tfrac{1}{2}\sqrt{[(x+y)^2 - z^2][z^2 - (x-y)^2]}\\ &= \tfrac{1}{2}\sqrt{2(x^2y^2 + x^2z^2 + y^2z^2) - x^4 - y^4 - z^4}.\end{aligned} \tag{2.4}$$

Observe that $\mathcal{K}_L^{(\alpha)}$ is a symmetric function of its three variables.

In the limit $\alpha \to -1/2$, identity (2.3) reduces to

$$\begin{aligned}\mathcal{L}_n^{(-1/2)}(x)\,\mathcal{L}_n^{(-1/2)}(y) &= \tfrac{1}{2}\{\mathcal{L}_n^{(-1/2)}(|x-y|) + \mathcal{L}_n^{(-1/2)}(x+y)\}\\ &\quad - \tfrac{1}{4}\int_{|x-y|}^{x+y} \mathcal{L}_n^{(-1/2)}(z)[xyz\,\mathcal{J}_1(\Delta)]\,dz.\end{aligned} \tag{2.5}$$

Let $\mu > 0$. In order to derive the Hermite product formula in the form (1.5)–(1.6), we first distinguish between even and odd indices. In the first case it follows from (2.2)–(2.3) that for any $k \in \mathbb{N}_0$, $x, y \in \mathbb{R}$,

$$\begin{aligned}\mathcal{H}_{2k}^{(\mu)}(x)\,\mathcal{H}_{2k}^{(\mu)}(y) &= \mathcal{L}_k^{(\mu-1/2)}(|x|)\,\mathcal{L}_k^{(\mu-1/2)}(|y|)\\ &= \int_{||x|-|y||}^{|x|+|y|} \mathcal{L}_k^{(\mu-1/2)}(z)\,\mathcal{K}_L^{(\mu-1/2)}(|x|, |y|, z)\,z^{2\mu}\,dz.\end{aligned}$$

By substituting $z \to -z$, the integral turns into

$$\int_{-|x|-|y|}^{-||x|-|y||} \mathcal{L}_k^{(\mu-1/2)}(|z|)\,\mathcal{K}_L^{(\mu-1/2)}(|x|, |y|, |z|)\,|z|^{2\mu}\,dz.$$

Hence, a linear combination of both integrals yields a product formula with symmetric support (1.7), i.e.,

$$\begin{aligned}\mathcal{H}_{2k}^{(\mu)}(x)\,\mathcal{H}_{2k}^{(\mu)}(y) &= \tfrac{1}{2}\int_{S(x,y)} \mathcal{H}_{2k}^{(\mu)}(z)\,\mathcal{K}_L^{(\mu-1/2)}(|x|, |y|, |z|)\,|z|^{2\mu}\,dz\\ &=: \int_{-\infty}^{\infty} \mathcal{H}_{2k}^{(\mu)}(z)\,dm_{x,y}^{e}(z).\end{aligned} \tag{2.6}$$

Similarly, in the odd order case, one obtains that

$$\begin{aligned}&\mathcal{H}_{2k+1}^{(\mu)}(x)\,\mathcal{H}_{2k+1}^{(\mu)}(y)\\ &\quad = xy\,\mathcal{L}_k^{(\mu+1/2)}(|x|)\,\mathcal{L}_k^{(\mu+1/2)}(|y|)\\ &\quad = \int_{||x|-|y||}^{|x|+|y|} \mathcal{L}_k^{(\mu+1/2)}(z)\,\mathcal{K}_L^{(\mu+1/2)}(|x|, |y|, z)\,xy\,z^{2\mu+2}\,dz\\ &\quad = \int_{-|x|-|y|}^{-||x|-|y||} z\mathcal{L}_k^{(\mu+1/2)}(|z|)\,\mathcal{K}_L^{(\mu+1/2)}(|x|, |y|, |z|)\,xyz\,|z|^{2\mu}\,dz,\end{aligned}$$

so that

$$\mathscr{H}_{2k+1}^{(\mu)}(x)\,\mathscr{H}_{2k+1}^{(\mu)}(y)$$
$$=\tfrac{1}{2}\int_{S(x,y)}\mathscr{H}_{2k+1}^{(\mu)}(z)\,\mathscr{K}_L^{(\mu+1/2)}(|x|,|y|,|z|)\,xyz\,|z|^{2\mu}\,dz$$
$$=:\int_{-\infty}^{\infty}\mathscr{H}_{2k+1}^{(\mu)}(z)\,dm_{x,y}^{o}(z). \tag{2.7}$$

But since, by construction,

$$\int_{-\infty}^{\infty}\mathscr{H}_{2k+1}^{(\mu)}(z)\,dm_{x,y}^{e}(z)=\int_{-\infty}^{\infty}\mathscr{H}_{2k}^{(\mu)}(z)\,dm_{x,y}^{o}(z)=0 \qquad (k\in\mathbb{N}_0),$$

the two identities (2.6)–(2.7) can be combined into

$$\mathscr{H}_n^{(\mu)}(x)\,\mathscr{H}_n^{(\mu)}(y)=\int_{-\infty}^{\infty}\mathscr{H}_n^{(\mu)}(z)\,\{dm_{x,y}^{e}(z)+dm_{x,y}^{o}(z)\}.$$

Returning to the Hermite polynomials via (2.1) again and observing that $\Delta(x, y, z)$ is an even function with respect to each variable, we obtain the following result.

THEOREM 2.1. *Let* $S(x, y)$, $\Delta$, $\mathscr{J}_\nu$, *and* $\mathscr{K}_L^{(\alpha)}$ *be as in* (1.7), (2.4), (1.9), *and* (2.3b), *respectively. For* $\mu>0$, $n\in\mathbb{N}_0$, *and* $x, y\in\mathbb{R}\setminus\{0\}$, *one has*

$$\tilde{H}_n^{(\mu)}(x)\,\tilde{H}_n^{(\mu)}(y)=\int_{-\infty}^{\infty}\tilde{H}_n^{(\mu)}(z)\,K_H^{(\mu)}(x,y,z)\,e^{-z^2}\,|z|^{2\mu}\,dz, \tag{2.8a}$$

*where*

$$K_H^{(\mu)}(x,y,z)=\begin{cases}\mathscr{K}_H^{(\mu)}(x,y,z)\,e^{(x^2+y^2+z^2)/2} & \text{if } z\in S(x,y)\\ 0 & \text{if } z\in\mathbb{R}\setminus S(x,y),\end{cases} \tag{2.8b}$$

*and*

$$\mathscr{K}_H^{(\mu)}(x,y,z)$$
$$=\frac{1}{2}\{\mathscr{K}_L^{(\mu-1/2)}(|x|,|y|,|z|)+\mathscr{K}_L^{(\mu+1/2)}(|x|,|y|,|z|)\,xyz\}$$
$$=\frac{1}{2}\frac{\Gamma(\mu+1/2)}{\Gamma(\mu)\,\Gamma(1/2)}\frac{\Delta^{2\mu-2}}{|xyz|^{2\mu-1}}\left\{\mathscr{J}_{\mu-1}(\Delta)+\frac{2\mu+1}{2\mu}\frac{\Delta^2}{xyz}\mathscr{J}_\mu(\Delta)\right\}. \tag{2.8c}$$

*Remark* 2.2. If $\mu$ is half of an odd integer, the "spherical" Bessel function $\mathscr{J}_\nu$ is given as a linear combination of sine and cosine functions with rational coefficients. In the first and simplest case $\mu = 1/2$, for example, the Bessel functions

$$\mathscr{J}_{-1/2}(z) = \cos z, \qquad \mathscr{J}_{1/2}(z) = \frac{\sin z}{z}$$

occur in the representation of the kernel (2.8b), so that

$$K_H^{(1/2)}(x, y, z) = \frac{1}{2\pi}\left\{\frac{\cos \Delta}{\Delta} + \frac{2 \sin \Delta}{xyz}\right\} e^{(x^2+y^2+z^2)/2}, \qquad z \in S(x, y).$$

THEOREM 2.3. *In the limiting case $\mu = 0$, the product formula of the classical Hermite polynomials $\tilde{H}_n := \tilde{H}_n^{(0)}$, $n \in \mathbb{N}_0$, is given for any $x, y \in \mathbb{R}$ by*

$$\begin{aligned}\tilde{H}_n(x)\,\tilde{H}_n(y) &= \tfrac{1}{4}\{[\tilde{H}_n(-x-y) + \tilde{H}_n(x+y)]\, e^{-xy} + [\tilde{H}_n(y-x) + \tilde{H}_n(x-y)]\, e^{xy}\} \\ &\quad + \int_{S(x,y)} \tilde{H}_n(z)\, K_H^{(0)}(x, y, z)\, e^{-z^2}\, dz, \qquad (2.9\text{a})\end{aligned}$$

*where*

$$K_H^{(0)}(x, y, z) = \left[-\frac{xyz}{8}\mathscr{J}_1(\Delta) + \frac{1}{4}\mathscr{J}_0(\Delta)\right] \operatorname{sgn}(xyz)\, e^{(x^2+y^2+z^2)/2}. \qquad (2.9\text{b})$$

*Proof.* Formula (2.9) follows either by taking the respective limit of identity (2.8) in a distributional sense or by employing directly the Laguerre product formula (2.5) and then proceeding as above. ∎

COROLLARY 2.4. *There is no $\mu \geqslant 0$ such that $K_H^{(\mu)}(x, y, z)$ is non-negative for all $x, y \in \mathbb{R}\setminus\{0\}$ and $z \in S(x, y)$.*

*Proof.* Suppose to the contrary that for some $\mu > 0$ one has $K_H^{(\mu)}(x, y, z) \geqslant 0$, $z \in S(x, y)$. By definition of $\mathscr{K}_H^{(\mu)}$ and $\mathscr{K}_L^{(\alpha)}$ and in view of the symmetry of $S(x, y)$, this would imply that

$$\begin{aligned}0 &\leqslant \mathscr{K}_H^{(\mu)}(x, y, z) + \mathscr{K}_H^{(\mu)}(x, y, -z) \\ &= \mathscr{K}_L^{(\mu-1/2)}(|x|, |y|, |z|) \\ &= \frac{\Gamma(\mu+1/2)}{\Gamma(\mu)\,\Gamma(1/2)} \frac{\Delta^{2\mu-2}}{|xyz|^{2\mu-1}} \mathscr{J}_{\mu-1}(\Delta).\end{aligned}$$

But since on the interval $||x|-|y|| \leqslant |z| \leqslant |x|+|y|$, the maximum of $\Delta(x, y, z)$ is attained for $z^2 = x^2 + y^2$ and given by $|xy|$, we obtain a contradiction if $|xy|$ is bigger than the smallest positive zero of the Bessel function $\mathscr{J}_{\mu-1}$.

The case $\mu=0$ is treated analogously. ∎

## 3. Norm Estimates of the Generalized Translation

The following estimates of the Bessel functions will be used in the sequel. If $\nu \geqslant -1/2$, the two upper bounds

$$|\mathscr{J}_\nu(x)| \leqslant \begin{cases} 1 \\ c_\nu \dfrac{\Gamma(\nu+1)}{\Gamma(1/2)}\left(\dfrac{2}{x}\right)^{\nu+1/2}, & x>0, \end{cases} \tag{3.1}$$

hold simultaneously, where $c_\nu=1$ if $|\nu| \leqslant 1/2$ and $c_\nu>1$ otherwise (cf. [19, 3.31(1); 18, Thm.7.31.2]). They are sharp for small and big values of $x$, respectively. For $-1<\nu<-1/2$, however, it follows from [19, 3.31(2)] and the asymptotic behavior of the Bessel function [19, 7.21(1)]

$$\mathscr{J}_\nu(x)=\frac{\Gamma(\nu+1)}{\Gamma(1/2)}\left(\frac{2}{x}\right)^{\nu+1/2}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+\mathcal{O}(x^{-\nu-3/2}) \qquad (x\to\infty),$$

that the two bounds on the right of (3.1) have to be combined to

$$|\mathscr{J}_\nu(x)| \leqslant c_\nu\left\{1+\left(\frac{2}{x}\right)^{\nu+1/2}\right\}, \qquad x>0, \quad -1<\nu<-1/2, \tag{3.2}$$

for some positive constant $c_\nu$.

In order to show that the generalized translation operator (1.11) as well as the convolution product (1.13) are well-defined on weighted Lebesgue spaces of the form (1.10), we have to estimate the $L^1_\mu$ norm of the kernel $K_H^{(\mu)}(\circ, y, z)$. To this end, the following two technical lemmas are required.

**Lemma 3.1.** *Let $b \leqslant 0$, $a+b>-1$, and let $\Delta$ be as in (2.4). For all $y, z>0$, one has*

$$\begin{aligned} I(a,b) &:= \int_{|y-z|}^{y+z} [\Delta(x,y,z)]^{2a}\, x^{2b+1}\,dx \\ &= \gamma(yz)^{2a+1}(y+z)^{2b}, \end{aligned} \tag{3.3}$$

*where*

$$\frac{\Gamma(a+1)\,\Gamma(1/2)}{\Gamma(a+3/2)} \leqslant \gamma \leqslant 2^{-b}\,\frac{\Gamma(a+1)\,\Gamma(1/2)}{\Gamma(a+b/2+3/2)}\,\frac{\Gamma(a+b+1)}{\Gamma(a+b/2+1)}.$$

*Proof.* Substituting $x^2 = \xi(\zeta) = (y+z)^2 - 4yz\zeta$ in the integral (3.3) and employing Euler's integral representation of the hypergeometric function [6, 2.1.3(10)], one obtains

$$\begin{aligned} I(a,b) &= 2^{-2a-1} \int_{(y-z)^2}^{(y+z)^2} ([y+z)^2 - \xi][\xi - (y-z)^2])^a\, \xi^b\, d\xi \\ &= (2yz)^{2a+1}\,(y+z)^{2b} \int_0^1 \zeta^a (1-\zeta)^a \left(1 - \frac{4yz}{(y+z)^2}\,\zeta\right)^b d\zeta \\ &= (2yz)^{2a+1}\,(y+z)^{2b}\,\frac{\Gamma^2(a+1)}{\Gamma(2a+2)}\,F\left(-b, a+1; 2a+2; \frac{4yz}{(y+z)^2}\right). \end{aligned}$$

By assumption, all three parameter entries of $F$ are non-negative, so that, in view of Gauß' Theorem,

$$\begin{aligned} 1 &\leqslant F\left(-b, a+1; 2a+2; \frac{4yz}{(y+z)^2}\right) \\ &\leqslant F(-b, a+1; 2a+2; 1) = \frac{\Gamma(2a+2)\,\Gamma(a+b+1)}{\Gamma(a+1)\,\Gamma(2a+b+2)}. \end{aligned}$$

An appeal to the duplication formula for the gamma function then yields the required estimates of the constant $\gamma$. ∎

LEMMA 3.2. *Let $\mathscr{K}_L^{(\alpha)}(x, y, z)$ for $\alpha > -1/2$ be given as in* (2.3b). *There are constants $D_1, D_2$, depending only on $\mu$, such that for any $y, z > 0$,*

$$\begin{aligned} A_1^{(\mu)}(y,z) &:= \int_{|y-z|}^{y+z} |\mathscr{K}_L^{(\mu-1/2)}(x,y,z)|\, x^{2\mu}\, dx \\ &\leqslant \begin{cases} 1 & \text{if } \mu \geqslant 1/2 \\ D_1\{1 + (yz)^{1/2-\mu}\} & \text{if } 0 < \mu < 1/2, \end{cases} \end{aligned} \tag{3.4}$$

$$\begin{aligned} A_2^{(\mu)}(y,z) &:= \int_{|y-z|}^{y+z} |\mathscr{K}_L^{(\mu+1/2)}(x,y,z)|\, x^{2\mu+1}\, dx \\ &\leqslant D_2 (yz)^{-1}, \qquad \mu \geqslant 0. \end{aligned} \tag{3.5}$$

*Proof.* By definition of $\mathscr{K}_L^{(\alpha)}$, we have

$$A_1^{(\mu)}(y,z) = C_{\mu-1/2}(yz)^{1-2\mu} \int_{|y-z|}^{y+z} \Delta^{2\mu-2}\, |\mathscr{J}_{\mu-1}(\Delta)|\, x\, dx.$$

If $\mu \geqslant 1/2$, the modulus of the Bessel function is uniformly bounded by 1 so that by Lemma 3.1,

$$A_1^{(\mu)}(y, z) \leqslant \frac{\Gamma(\mu+1/2)}{\Gamma(\mu)\,\Gamma(1/2)} (yz)^{1-2\mu}\, I(\mu-1, 0) \equiv 1.$$

If $0 < \mu < 1/2$, we use the estimate (3.2) to get

$$\begin{aligned} A_1^{(\mu)}(y, z) &\leqslant C_{\mu-1/2}(yz)^{1-2\mu}\, c_{\mu-1} \int_{|y-z|}^{y+z} \Delta^{2\mu-2}\left[1+\left(\frac{2}{\Delta}\right)^{\mu-1/2}\right] x\, dx \\ &= C_{\mu-1/2}c_{\mu-1}(yz)^{1-2\mu}\left\{I(\mu-1, 0) + 2^{\mu-1/2}\, I\left(\frac{\mu}{2}-\frac{3}{4}, 0\right)\right\}. \end{aligned}$$

Here, Lemma 3.1 yields the second estimate in (3.4).

Concerning (3.5), we start off from

$$A_2^{(\mu)}(y, z) = C_{\mu+1/2}(yz)^{-2\mu-1} \int_{|y-z|}^{y+z} \Delta^{2\mu}\, |\mathcal{J}_\mu(\Delta)|\, dx.$$

Since $\mu \geqslant 0$, the two estimates in (3.1) can be applied alternatively to obtain

$$\begin{aligned} A_2^{(\mu)}(y, z) &\leqslant C_{\mu+1/2}(yz)^{-2\mu-1} \begin{cases} I\left(\mu, -\dfrac{1}{2}\right) \\ c_\mu \dfrac{\Gamma(\mu+1)}{\Gamma(1/2)}\, 2^{\mu+1/2}\, I\left(\dfrac{\mu}{2}-\dfrac{1}{4}; -\dfrac{1}{2}\right) \end{cases} \\ &\leqslant D_2 \frac{2}{y+z} \begin{cases} 1 \\ (yz)^{-\mu-1/2} \end{cases} \end{aligned}$$

for $yz \neq 0$ and for some positive constant $D_2$. Hence,

$$\begin{aligned} A_2^{(\mu)}(y, z) &\leqslant D_2(yz)^{-1} \min\left\{\frac{2yz}{y+z}, \frac{2(yz)^{1/2-\mu}}{y+z}\right\} \\ &\leqslant D_2(yz)^{-1} \min\{(yz)^{1/2}, (yz)^{-\mu}\} \\ &\leqslant D_2(yz)^{-1}. \quad ▮ \end{aligned}$$

So prepared we can state and prove the main result of this section.

THEOREM 3.3. (a) *The kernel* (2.8b) *of the generalized Hermite product formula satisfies the norm estimate*

$$e^{-(y^2+z^2)/2}\,\|K_H^{(\mu)}(\circ, y, z)\|_{1,\mu} \leqslant M_\mu \begin{cases} 1 & \text{if } \mu \geqslant 1/2 \\ 1+|yz|^{1/2-\mu} & \text{if } 0<\mu<1/2 \\ |yz|^{1/2} & \text{if } \mu=0 \end{cases} \tag{3.6}$$

*for* $yz \neq 0$, *where* $M_\mu \geqslant 1$ *denotes a constant depending only on* $\mu$.

(b) *If* $\mu \geqslant 1/2$, *the generalized translation* (1.11) *is a bounded linear operator from* $L_\mu^p$, $1 \leqslant p \leqslant \infty$, *into itself with*

$$\|T_y^{(\mu)}\|_{[p,\mu]} := \|T_y^{(\mu)}\|_{[L_\mu^p]} \leqslant M_\mu e^{y^2/2} \qquad (y \in \mathbb{R}). \tag{3.7}$$

*If* $0 \leqslant \mu < 1/2$, *one has for any* $f \in L_\mu^1 \cap L_{\mu/2+1/4}^1$ *that*

$$\|T_y^\mu f\|_{1,\mu} \leqslant M_\mu e^{y^2/2}\{\|f\|_{1,\mu} + |y|^{1/2-\mu}\,\|f\|_{1,\mu/2+1/4}\}. \tag{3.8}$$

*Proof.* (a) For any $\mu > 0$ and $yz \neq 0$, it follows from (2.8) and the symmetry of the support $S(y, z)$ that

$$\begin{aligned} e^{-(y^2+z^2)/2}\,&\|K_H^{(\mu)}(\circ, y, z)\|_{1,\mu} \\ &= \int_{S(y,z)} |\mathscr{K}_H^{(\mu)}(x, y, z)|\,|x|^{2\mu}\,dx \\ &\leqslant \tfrac{1}{2}\int_{S(y,z)} |\mathscr{K}_L^{(\mu-1/2)}(|x|, |y|, |z|)|\,|x|^{2\mu}\,dx \\ &\quad + \tfrac{1}{2}\int_{S(y,z)} |\mathscr{K}_L^{(\mu+1/2)}(|x|, |y|, |z|)|\,|x|^{2\mu+1}\,dx\,|yz| \\ &= A_1^{(\mu)}(|y|, |z|) + A_2^{(\mu)}(|y|, |z|)\,|yz|, \end{aligned}$$

$A_j^{(\mu)}$, $j = 1, 2$, being as in Lemma 3.2. In view of the estimates (3.4)–(3.5), the last expression has the upper bound

$$\begin{array}{ll} 1 + D_2 & \text{if } \mu \geqslant 1/2 \\ D_1\{1 + |yz|^{1/2-\mu}\} + D_2 & \text{if } 0 < \mu < 1/2 \end{array}.$$

In the limit case $\mu = 0$, we use directly the definition (2.9b) of $K_H^{(0)}$ to deduce

$$\begin{aligned} e^{-(y^2+z^2)/2}\,&\|K_H^{(0)}(\circ, y, z)\|_{1,0} \\ &= \int_{S(y,z)} \left| -\frac{xyz}{8}\mathscr{J}_1(\Delta) + \frac{1}{4}\mathscr{J}_0(\Delta)\right| dx \\ &\leqslant \frac{|yz|}{4}\int_{||y|-|z||}^{|y|+|z|} |\mathscr{J}_1(\Delta)|\,x\,dx + \frac{1}{2}\int_{||y|-|z||}^{|y|+|z|} |\mathscr{J}_0(\Delta)|\,dx. \end{aligned}$$

Here, the two bounds of the Bessel function, (3.1), together with (3.3) imply that the two terms in the last line are bounded from above by some constant times $\min\{|yz|^2, |yz|^{1/2}\}$ and $\min\{|yz|^{1/2}, 1\}$, respectively. This readily gives the third line of (3.6) and concludes the proof of part (a).

(b) The assertion is trivial in case $y=0$ by definition of $T_0$, so assume that $y \neq 0$. For $\mu \geqslant 1/2$, the norm estimate (3.7) is an immediate consequence of the first line of (3.6). Similarly, if $0<\mu<1/2$, the second line of (3.6) implies that

$$\| T_y^{(\mu)}(f; x)\|_{1,\mu} \leqslant \int_{-\infty}^{\infty} |f(z)|\, e^{-z^2} |z|^{2\mu} \, \| K_H^{(\mu)}(\circ, y, z)\|_{1,\mu}\, dz$$

$$\leqslant M_\mu e^{y^2/2} \int_{-\infty}^{\infty} |f(z)|\, e^{-z^2/2} |z|^{2\mu} [1 + |yz|^{1/1-\mu}]\, dz,$$

where the interchange of the order of integration is justified by Fubini's Theorem, provided that $f \in L_\mu^1 \cap L_{\mu/2+1/4}^1$. Finally, one has for $\mu=0$ that

$$\| T_y^{(0)}(f; x)\|_{1,0} \leqslant \tfrac{1}{4} e^{y^2/2} \int_{-\infty}^{\infty} [|f(-x-y)| + |f(x+y)|]\, e^{-(x+y)^2/2}\, dx$$

$$+ \tfrac{1}{4} e^{y^2/2} \int_{-\infty}^{\infty} [|f(x-y)| + |f(y-x)|]\, e^{-(x-y)^2/2}\, dx$$

$$+ \int_{-\infty}^{\infty} |f(z)|\, e^{-z^2} \, \| K_H^{(0)}(\circ, y, z)\|_{1,0}\, dz$$

$$\leqslant e^{y^2/2} \{\| f \|_{1,0} + M_0 |y|^{1/2} \| f \|_{1,1/4}\}.$$

Together this yields (3.8). ∎

For $p=1$, the norm estimate of the (generalized) Hermite translation, (3.7), is sharp in the following strong sense.

THEOREM 3.4. (a) *Let $\mu \geqslant 0$. For any $y \in \mathbb{R}$, one has*

$$\| T_y^{(\mu)} \|_{[1,\mu]} \geqslant e^{y^2/2}. \tag{3.9}$$

(b) *For any $y \neq 0$ there is a constant $\mu_0 = \mu_0(y) > 0$ such that*

$$\| T_y^{(\mu)} \|_{[1,\mu]} \geqslant \tfrac{1}{2} |y|\, e^{y^2/2}, \qquad \mu \geqslant \mu_0. \tag{3.10}$$

*In particular, for sufficiently large $\mu$, the constant $M_\mu$ on the right-hand side of* (3.7) *cannot be replaced by* 1.

*Proof.* The two estimates from below, (3.9)–(3.10), are obtained by applying the translation $T_y^{(\mu)}$ to appropriate test functions. To this end we choose the even and odd components of the generating function of the generalized Hermite polynomials, respectively. In view of (1.4) and the well-known generating function of the Laguerre polynomials [6, 10.12(17)], they are given by

$$\begin{aligned} f_s^e(x) &:= \sum_{k=0}^{\infty} [\tilde{h}_{2k}^{(\mu)}]^{-1} \tilde{H}_{2k}^{(\mu)}(x)\, s^{2k} \\ &= \frac{1}{\Gamma(\mu+1/2)} \sum_{k=0}^{\infty} L_k^{(\mu-1/2)}(x^2)(s^2)^k \\ &= \frac{1}{\Gamma(\mu+1/2)} (1-s^2)^{-\mu-1/2} \exp\left(\frac{x^2 s^2}{s^2-1}\right), \\ f_s^o(x) &:= \sum_{k=0}^{\infty} [\tilde{h}_{2k+1}^{(\mu)}]^{-1} \tilde{H}_{2k+1}^{(\mu)}(x)\, s^{2k+1} \\ &= \frac{xs}{\Gamma(\mu+3/2)} \sum_{k=0}^{\infty} L_k^{(\mu+1/2)}(x^2)(s^2)^k \\ &= \frac{xs}{\Gamma(\mu+3/2)} (1-s^2)^{-\mu-3/2} \exp\left(\frac{x^2 s^2}{s^2-1}\right) \end{aligned}$$

for $0 < s < 1$. Clearly, both functions belong to $L_\mu^1$. In fact, with

$$a(s) = \frac{1}{2} + \frac{s^2}{1-s^2} = \frac{1}{2}\frac{1+s^2}{1-s^2},$$

their norms are easily computed to

$$\begin{aligned} \| f_s^e \|_{1,\mu} &= \frac{2}{\Gamma(\mu+1/2)} (1-s^2)^{-\mu-1/2} \int_0^\infty e^{-ax^2} x^{2\mu}\, dx \\ &= (1-s^2)^{-\mu-1/2} a^{-\mu-1/2} = \left(\frac{2}{1+s^2}\right)^{\mu+1/2} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \| f_s^o \|_{1,\mu} &= \frac{2s}{\Gamma(\mu+3/2)} (1-s^2)^{-\mu-3/2} \int_0^\infty e^{-ax^2} x^{2\mu+1}\, dx \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+3/2)} \frac{s}{\sqrt{1-s^2}} \left(\frac{2}{1+s^2}\right)^{\mu+1}. \end{aligned} \tag{3.12}$$

Employing, on the other hand, the bilinear generating function of the Laguerre polynomials [6, 10.12(20)], one similarly obtains

$$T_y^{(\mu)}(f_s^e; x)$$
$$= \sum_{k=0}^{\infty} [\tilde{h}_{2k}^{(\mu)}]^{-1} \tilde{H}_{2k}^{(\mu)}(x)\, \tilde{H}_{2k}^{(\mu)}(y)\, s^{2k}$$
$$= \frac{1}{\Gamma(\mu+1/2)} \sum_{k=0}^{\infty} L_k^{(\mu-1/2)}(x^2)\, R_k^{(\mu-1/2)}(y^2)(s^2)^k$$
$$= \frac{1}{\Gamma(\mu+1/2)} (1-s^2)^{-\mu-1/2}\, \mathscr{I}_{\mu-1/2}\left(2\frac{xys}{1-s^2}\right) \exp\left(\frac{[x^2+y^2]\, s^2}{s^2-1}\right),$$
$$T_y^{(\mu)}(f_s^o; x)$$
$$= \sum_{k=0}^{\infty} [\tilde{h}_{2k+1}^{(\mu)}]^{-1} \tilde{H}_{2k+1}^{(\mu)}(x)\, \tilde{H}_{2k+1}^{(\mu)}(y)\, s^{2k+1}$$
$$= \frac{xys}{\Gamma(\mu+3/2)} \sum_{k=0}^{\infty} L_k^{(\mu+1/2)}(x^2)\, R_k^{(\mu+1/2)}(y^2)(s^2)^k$$
$$= \frac{xys}{\Gamma(\mu+3/2)} (1-s^2)^{-\mu-3/2}\, \mathscr{I}_{\mu+1/2}\left(2\frac{xys}{1-s^2}\right) \exp\left(\frac{[x^2+y^2]\, s^2}{s^2-1}\right).$$

Here $\mathscr{I}_\nu(z)$ denotes the (normalized) modified Bessel function of the first kind defined by

$$\mathscr{I}_\nu(z) = 2^\nu \Gamma(\nu+1)\, z^{-\nu} I_\nu(z) = {}_0F_1(-; \nu+1; z^2/4).$$

Notice that $\mathscr{I}_\nu$ is positive on the real line and that for $\gamma > 0$ and $\lambda, \nu > -1$,

$$2\int_0^\infty \mathscr{I}_\nu(2\beta x)\, x^{2\lambda+1} e^{-\gamma x^2}\, dx$$
$$= \Gamma(\lambda+1)\, \gamma^{-\lambda-1}\, {}_1F_1(\lambda+1; \nu+1; \beta^2/\gamma)$$
$$= \Gamma(\lambda+1)\, \gamma^{-\lambda-1} \exp(\beta^2/\gamma)\, {}_1F_1(\nu-\lambda; \nu+1; -\beta^2/\gamma). \tag{3.13}$$

Indeed, the first identity follows by inserting the power series representation of $\mathscr{I}_\nu$ under the integral and then integrating termwise, while the second one is a consequence of Kummer's transformation [6, 6.4(7)]. A short calculation then yields

$$\| T_y^{(\mu)} f_s^e \|_{1,\mu} = \frac{1}{\Gamma(\mu+1/2)} (1-s^2)^{\mu-1/2} \exp\left(\frac{(ys)^2}{s^2-1}\right)$$
$$\times 2\int_0^\infty \mathscr{I}_{\mu-1/2}\left(\frac{2ys}{1-s^2}x\right) x^{2\mu} e^{-a(s)x^2}\, dx$$
$$= \left(\frac{2}{1+s^2}\right)^{\mu+1/2} \exp\left(\frac{s^2}{1+s^2} y^2\right).$$

Dividing this expression by the norm of the test function $f_s^e$, (3.11), and letting $s$ tend to $1-$, we arrive at the required estimate (3.9), since

$$\| T_y^{(\mu)} \|_{[1,\mu]} \geq \sup_{0 \leq s < 1} \{ \| T_y^{(\mu)} f_s^e \|_{1,\mu} / \| f_s^e \|_{1,\mu} \}$$

$$= \sup_{0 \leq s < 1} \left\{ \exp\left( \frac{s^2}{1+s^2} y^2 \right) \right\} = \exp(y^2/2).$$

As to the second estimate (3.10), an application of formula (3.13) with $\beta = ys/(1-s^2)$, $\nu = \mu + 1/2$, $\lambda = \mu$, and $\gamma = a(s)$ yields

$$\| T_y^{(\mu)} f_s^o \|_{1,\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+3/2)} \, |ys| (1-s^2)^{-\mu-3/2} \left( \frac{1}{2} \frac{1+s^2}{1-s^2} \right)^{-\mu-1}$$

$$\cdot \exp\left( \frac{2(ys)^2}{1-s^4} - \frac{(ys)^2}{1-s^2} \right) {}_1F_1\left( \frac{1}{2}; \mu + \frac{3}{2}; -\frac{2(sy)^2}{1-s^4} \right).$$

In view of (3.12), this gives

$$\| T_y^{\mu} \|_{[1,\mu]} \geq \sup_{0 \leq s < 1} \left\{ \| T_y^{(\mu)} f_s^o \|_{1,\mu} / \| f_s^o \|_{1,\mu} \right\}$$

$$= \sup_{0 \leq s < 1} \left\{ |y| \exp\left( \frac{(ys)^2}{1+s^2} \right) {}_1F_1\left( \frac{1}{2}; \mu + \frac{3}{2}; -\frac{2(sy)^2}{1-s^4} \right) \right\}.$$

If we choose now

$$s^2 = \sigma(y, \mu) := \sqrt{1 + y^4/\mu^2} - y^2/\mu, \qquad y \neq 0, \quad \mu > 0,$$

which is clearly in (0, 1), we find that

$$\| T_y^{\mu} \|_{[1,\mu]} \geq |y| \exp\left( \frac{\sigma}{1+\sigma} y^2 \right) {}_1F_1\left( \frac{1}{2}; \mu + \frac{3}{2}; -\mu \right).$$

Observing finally that $\sigma(y, \mu) \to 1-$ as $\mu \to \infty$, and that the ${}_1F_1$ function approaches $1/\sqrt{2}$ (cf. [6, 6.13.3]), we obtain inequality (3.10) for $\mu$ sufficiently large. This completes the proof of Theorem 3.4. ∎

The estimates (3.7)–(3.8) of the generalized Hermite translation operator, in turn, are suited to obtain certain norm inequalities for the convolution product (1.13) which lead to the following result including (1.14).

THEOREM 3.5. (a) *Let* $\mu \geq 1/2$ *and* $1 \leq p \leq \infty$. *If* $f \in L_\mu^p$ *and* $g \in L_\mu^1$, *then* $f * g \in L_\mu^p$.

(b) *For* $0 \leq \mu < 1/2$, $f * g \in L_\mu^1$ *if* $f, g \in L_\mu^1 \cap L_{\mu/2+1/4}^1$.

*Proof.* We first note that $T_y^{(\mu)}(f;x)=T_x^{(\mu)}(f;y)$ in view of the symmetry of the measure $m_{x,y}$ in $x$ and $y$.

(a) Since $\mu \geqslant 1/2$, we can apply (3.7) to find

$$\|f*g\|_{p,\mu} \leqslant \int_{-\infty}^{\infty} \|T_x^{(\mu)}(f;\circ)\|_{p,\mu} |g(x)| e^{-x^2} |x|^{2\mu}\,dx$$
$$\leqslant M_\mu \|f\|_{p,\mu} \|g\|_{1,\mu},$$

provided that $f\in L_\mu^p$ and $g\in L_\mu^1$.

(b) If $0\leqslant \mu<1/2$, it follows from (3.8) that for any $f$ and $g$ which simultaneously belong to $L_\mu^1$ and $L_{\mu/2+1/4}^1$,

$$\|f*g\|_{1,\mu} \leqslant \int_{-\infty}^{\infty} \|T_x^{(\mu)}(f,\circ)\|_{1,\mu} |g(x)| e^{-x^2} |x|^{2\mu}\,dx$$
$$\leqslant M_\mu \{\|f\|_{1,\mu} \|g\|_{1,\mu} + \|f\|_{1,\mu/2+1/4} \|g\|_{1,\mu/2+1/4}\}. \quad \blacksquare$$

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